Positivity of energy in five dimensional classical unified field theories, II*

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Abstract. *Five dimensional classical unified field theories as well as Yang-Mills theory with gauge group $U(1)$, are described in terms of a Lorentzian five dimensional space V_5 with metric tensor $\gamma_{\alpha\beta}$ which admits a space-like Killing vector ζ^α. It is assumed that:* (1) *V_5 has the topology of $V_4 \times S^1$, S^1 is a circle and V_4 is a four dimensional Lorentzian space that is asymptotically flat and* (2) *the Einstein tensor $\Gamma_{\alpha\beta}$ of V_5 satisfies $\Gamma_{\alpha\beta}U^\alpha v^\beta \leqslant 0$ where U^α and v^α are future oriented time-like vectors with $\gamma_{\alpha\beta}v^\alpha\zeta^\beta = 0$. The spinor approach of Witten* [4], *Nester* [3] *and Moreschi and Sparling* [5] *is used to show that the conserved five dimensional energymomentum vector P^α is non-space-like. If $P^\alpha = \Gamma_{\alpha\beta} = 0$ then V_5 must admit a time-like Killing vector. Lichnerowicz's results* [1] *then imply that V_5 must be flat. A lower bound for P^4 (the mass) similar to that found by Gibbons and Hull* [6] *is obtained.*

INTRODUCTION

It is the purpose of this paper to prove the analogue of the positive energy theorem of the Einstein theory of general relativity for a five dimensional space V_5 that has a Lorentzian metric $\gamma_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3, 4, 5$ throughout the paper) and which admits a space like Killing vector ζ^α. We may choose coordinates in V_5 so that

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$$ds^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta = g_{ij} dx^i dx^j + W^2(dx^5 + A_i dx^i)^2, \tag{1.1}$$

where i, j take on values from 1 to 4

$$\zeta^\alpha = \delta_5^\alpha, \tag{1.2}$$

and

$$\zeta^\alpha \zeta^\beta \gamma_{\alpha\beta} = W^2 \geqslant 0. \tag{1.3}$$

The quantities $g_{i,j}$, A_i and W are independent of x^5 and are functions of the x^i. They are determined from field equations that are derived in various theories in which the space V_5 plays a role.

This space may be considered as the principal bundle over a four dimensional space with metric tensor g_{ij} with fibre group $U(1)$. The A_i are then the connection of this bundle and the $\gamma_{\alpha\beta}$ form the coefficients of the right translational invariant metric on the bundle. W is then a scalar field over V_4. We shall restrict the bunlde to be a trivial one. That is we consider the case where V_5 is diffeomorphic to $V_4 \times S^1$, (S^1 being a circle).

The space V_5 arises in the classical unified field theories of Jordan-Thiry and of Kaluza-Klein [1] and in Veblen's theory of projective relativity [2]. In the two latter theories it is assumed that $W = 1$. The geometric interpretation of A_i differs from that stated above however in all of the theories mentioned it is interpreted physically as being proportional to the potential vector of a Maxwell field. In the Jordan-Thiry theory the scalar field is interpreted as being related to a varying gravitational «constant».

We shall take as allowable coordinate transformations those which preserve equations (1.3). These are given by equations of the form

$$\begin{aligned} \bar{x}^i &= \bar{x}^i(x^i) \\ \bar{x}^5 &= x^5 + \Lambda(x^i). \end{aligned} \tag{1.4}$$

Under such a transformation we have

$$\begin{aligned} \bar{g}_{ij} &= g_{kl} \frac{\partial x^k}{\partial x^i} \frac{\partial \bar{x}^l}{\partial x^j} \\ \bar{A}_i &= A_j \frac{\partial x^j}{\partial \bar{x}^i} - \Lambda_{,i} \\ \bar{A}_5 &= 1 \\ \bar{W}(\bar{x}^i) &= W(x^i(\bar{x})) \end{aligned} \tag{1.5}$$

where we use the notation that

$$\Lambda_{,i} = \frac{\partial \Lambda}{\partial x^i} \tag{1.6}$$

We shall assume that the space V_4 admits spinors and that these can be lifted to the space V_5. We shall use a slight modification of Nester's method [3] for dealing with Witten's [4] proof of the positive energy theorem in V_4. The modification has been used by Moreschi and Sparling [5] and consists of using a three-form in V_5 formed from four-component spinor field and its covariant derivatives instead of a two-form in V_4 formed similarly.

The positive energy theorem given below is similar to that of Gibbons and Hull [6]. It is derived by assuming the existence of solutions of the analogue of Witten's equation in a four-dimensional space-like hypersurface in V_5. This equation may also be written as an equation in a three-dimensional hypersurface in V_4. The existence of asymptotically constant solutions of this equation has been proven by T.H. Parker [7].

2. THE VACUUM FIELD EQUATIONS

These equations are obtained from the variational principle

$$\delta I = 0$$

where

$$I = \int \sqrt{-\gamma}\, B \mathrm{d}^5 x. \tag{2.1}$$

B is the scalar curvature formed from the $\gamma_{\alpha\beta}$ and all the $\gamma_{\alpha\beta}$ are varied subject to the condition that they and their partial derivatives vanish on the boundary of the region of integration. The Euler equations of this principle are then

$$\Gamma^{\alpha\beta} = B^{\alpha\beta} - \frac{1}{2}\gamma^{\alpha\beta} B = 0 \tag{2.2}$$

where $B^{\alpha\beta}$ is the Ricci tensor of V_5.

If the variation is subject to the condition that

$$\delta\gamma_{55} = \delta(W^2) = 0$$

as is the case in the Klein-Kaluza theory and in projective relativity then the Euler equations are

(2.3) $$\Gamma_{\alpha\beta} - A_\alpha A_\beta \Gamma = 0.$$

We denote the covariant derivative of a X_α in V_5 by $X_{\alpha;\beta}$. We further define the Riemann-Christoffel curvature tensor of V_5, $B^{\delta}_{\alpha\beta\gamma}$ to be such that

(2.4) $$X_{\alpha;\beta\gamma} - X_{\alpha;\gamma\beta} = X_\delta B^{\delta}_{\alpha\beta\gamma}.$$

The Ricci tensor is taken to be

(2.5) $$B_{\alpha\beta} = B^{\delta}_{\alpha\beta\delta}.$$

In the discussion of the geometry of V_5 it is convenient to deconpose various indexed quantities into those with no index equal to 5, with one index equal to five, with two indices equal to five etc. Thus we have

(2.6)
$$\begin{aligned}
g_{ij} &= \gamma_{ij} - \frac{\gamma_{5i}\gamma_{5j}}{\gamma_{55}} \\
\gamma_{5i} &= W^2 A_i \\
\gamma_{55} &= W^2 \\
\gamma^{ij} &= g^{ij} \\
\gamma^{5i} &= -g^{ij} A_j = -A^i \\
\gamma^{55} &= W^{-2} + g^{ij} A_i A_j \\
\gamma &= \det(\gamma_{\alpha\beta}) = W^2 \det(g_{ij})
\end{aligned}$$

where $g^{ij} g_{jk} = \delta^i_k$.

Let $\omega^{\underline{a}} (a = 1, 2, 3, 4)$ be a set of four one forms in V_5 such that

(2.7) $$\omega^{\underline{a}}_5 = 0$$

and

(2.8) $$\eta_{\underline{ab}}\, \omega^{\underline{a}} \omega^{\underline{b}} = g_{ij}\, \mathrm{d}x^i \mathrm{d}x^j$$

where $\eta_{\underline{ab}}$ are the components of the metric tensor in Minkowski space with signature $(+, +, +, -)$.

Let

$$\omega^{\underline{5}} = W(A_i \mathrm{d}x^i + \mathrm{d}x^5)$$

so that

(2.9) $$\omega^{\underline{5}}_i = W A_i, \qquad \omega^{\underline{5}}_i = W.$$

Then

$$\gamma_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta = \eta_{\underline{ab}} \omega^{\underline{a}} \omega^{\underline{b}} + (\omega^{\underline{5}})^2 = \eta_{\underline{\alpha\beta}} \omega^{\underline{\alpha}} \omega^{\underline{\beta}}.$$

The five one-forms determine an orthonormal pentad of vectors e_α such that

$$\omega^{\underline{\alpha}}(e_{\underline{\beta}}) = \delta^\alpha_\beta \qquad (\underline{\alpha}, \underline{\beta} = 1, 2, 3, 4, 5)$$

we have

$$(2.10) \qquad \begin{aligned} &e^i_{\underline{5}} = 0, \quad e^5_{\underline{5}} = W^{-1}, && e^5_{\underline{b}} = -A_j e^j_{\underline{b}} = -A_{\underline{b}}, \\ &e^i_{\underline{b}} \omega^{\underline{a}}_i = \delta^{\underline{a}}_{\underline{b}}, && e^j_{\underline{a}} \omega^{\underline{5}}_j = W A_{\underline{a}}. \end{aligned}$$

The Cartan connection one forms $\omega^{\underline{a}}{}_{\underline{b}}$ are defined by the equations

$$(2.11) \qquad \mathrm{d}\omega^{\underline{a}} = \omega^{\underline{b}} \wedge \omega^a{}_{\underline{b}}$$

where as usual d denotes the exterior derivative and $\wedge$ denotes the exterior product. If may be verified that

$$(2.12) \qquad \begin{aligned} &\omega^{\underline{a}}{}_{\underline{b}i} = \tilde{\omega}^{\underline{a}}{}_{\underline{b}i} + F^{\underline{a}}{}_{\underline{b}} \omega^5_i \\ &\omega^{\underline{a}}{}_{\underline{b}5} = W F^{\underline{a}}{}_{\underline{b}} \\ &\omega^{\underline{a}}{}_{\underline{5}i} = F^{\underline{a}}{}_{\underline{b}} \omega^{\underline{b}}{}_i - \rho^{\underline{a}} \omega^{\underline{5}}_i \\ &\omega^{\underline{a}}{}_{\underline{5}5} = -W\rho^{\underline{a}} \end{aligned}$$

where

$$(2.13) \qquad \begin{aligned} &\rho_{\underline{b}} = \frac{W_{,i}}{W} e^i_{\underline{b}} \\ &F_{\underline{ab}} = \frac{W}{2} (A_{i,j} - A_{j,i}) e^i_{\underline{a}} e^j_{\underline{b}} \end{aligned}$$

and the underscored latin indices are manipulated by $\eta_{\underline{ab}}$ and $\eta^{\underline{ab}}$. In addition $\tilde{\omega}^{\underline{a}}_{\underline{b}i}$ are the connection one forms for V_4 with metric g_{ij} and are determined from the $\omega^{\underline{a}}_i$ by the analogue of equations (2.11).

The curvature two forms are given by the equations

$$B^{\underline{\alpha}}{}_{\underline{\beta}} = \mathrm{d}\omega^{\underline{\alpha}}{}_{\underline{\beta}} + \omega^{\underline{\alpha}}{}_{\underline{\gamma}} \wedge \omega^{\underline{\gamma}}{}_{\underline{\beta}}.$$

It may be shown that the components of the Ricci tensor of V_5 in the orthonormal frame are

$$(2.14) \qquad B_{\alpha\beta} e^\alpha_a e^\beta_b = B_{\underline{ab}} = R_{\underline{ab}} + \frac{W_{[\underline{ab}]}}{W} + 2F_{\underline{ac}} F_{\underline{b}}{}^{\underline{c}}$$

$$
(2.14)\qquad
\begin{aligned}
B_{\alpha\beta}e^{\alpha}_{\underline{a}}e^{\beta}_{\underline{5}} = B_{a5} &= \frac{1}{W^2}\,(W^2 F_{\underline{a}}^{\underline{c}})_{|\underline{c}} \\
B_{\alpha\beta}e^{\alpha}_{\underline{5}}e^{\beta}_{\underline{5}} = B_{\underline{55}} &= \frac{1}{W}\,\Delta W - F_{\underline{ab}}F^{\underline{ab}}
\end{aligned}
$$

and

$$
B = \eta^{\underline{\alpha\beta}}B_{\underline{\alpha\beta}} = R + \frac{2\Delta W}{W} + F_{\underline{ab}}F^{\underline{ab}}
$$

where

$$
(2.15)\qquad
\begin{aligned}
R_{\underline{ab}} &= R_{ij}e^{i}_{\underline{a}}e^{j}_{\underline{b}} \\
R &= \eta^{\underline{ab}}R_{\underline{ab}},
\end{aligned}
$$

R_{ij} is the Ricci tensor formed from g_{ij}, and the stroke denotes the covariant derivative with respect to this metric. Thus for example

$$
\phi_{\underline{a}|\underline{c}} = (\phi_{\underline{a},i} - \phi_{\underline{b}}\,\omega^{\underline{b}}{}_{\underline{a}i})\,e^{i}{}_{\underline{c}} = \phi_{i|j}e^{i}_{\underline{a}}e^{j}_{\underline{b}}
$$

and

$$
(2.16)\qquad
\begin{aligned}
W_{|\underline{ab}} &= W_{|ij}e^{i}_{\underline{a}}e^{j}_{\underline{b}} \\
F_{\underline{a}}^{\underline{c}}{}_{|\underline{c}} &= F^{j}_{i}{}_{|j}e^{i}_{\underline{a}} \\
\Delta W &= \eta^{\underline{ab}}W_{|\underline{ab}} = g^{ij}W_{|ij}.
\end{aligned}
$$

It follows from equations (2.2) and (2.14) that

$$
(2.17)\qquad
\begin{aligned}
\Gamma_{\underline{ab}} &= G_{\underline{ab}} + 2\left[F_{\underline{ac}}F_{\underline{b}}{}^{\underline{c}} - \frac{1}{4}\,\eta_{\underline{ab}}F^{\underline{cd}}F_{\underline{cd}}\right] + \frac{1}{W}\,[W_{|\underline{ab}} - \eta_{\underline{ab}}\Delta W] \\
\Gamma_{\underline{a5}} &= \frac{1}{W^2}\,(W^2 F_{\underline{a}}^{\underline{c}})_{|c} \\
\Gamma_{\underline{55}} &= -\frac{1}{2}\,(R + 3F^{\underline{ab}}F_{\underline{ab}})
\end{aligned}
$$

where

$$
(2.18)\qquad G_{\underline{ab}} = R_{\underline{ab}} - \frac{1}{2}\,\eta_{\underline{ab}}R.
$$

When we write

(2.19) $$A_i = (16\pi G)^{1/2}\phi_i$$

and define

(2.20) $$f_{ij} = \phi_{i,j} - \phi_{j,i}$$

(2.21) $$T^E_{\underline{ab}} = f_{\underline{ac}} f_{\underline{b}}{}^{\underline{c}} - \frac{1}{4}\eta_{\underline{ab}} f^{\underline{cd}} f_{\underline{cd}}$$

then the first of equations (2.17) becomes

(2.22) $$\Gamma_{\underline{ab}} = G_{\underline{ab}} + 8\pi G W^2 T^E_{\underline{ab}} + \frac{1}{W}[W_{|\underline{ab}} - \eta_{\underline{ab}}\Delta W]$$

T^E_{ab} is the stress-energy tensor of the Maxwell electromagnetic field determined by the potential ϕ_i. The second of equations (2.17) become

(2.23) $$\Gamma_{\underline{a5}} = \frac{(16\pi G)^{1/2}}{2}[W f_{\underline{a}}{}^{\underline{c}}{}_{|\underline{c}} + 3 W_{|\underline{c}} f_{\underline{a}}{}^{\underline{c}}].$$

When $W = 1$, the vacuum field equations for V_5, the equations $\Gamma_{\alpha\beta} - A_\alpha A_\beta \Gamma = 0$, become

(2.24) $$G_{\underline{ab}} = -8\pi G T^E_{\underline{ab}}$$

(2.25) $$f_{\underline{a}}{}^{\underline{c}}{}_{|\underline{c}} = 4\pi J_{\underline{a}} = 0.$$

These are the Einstein-Maxwell equations where the source of the gravitational filed is a source free electromagnetic field.

3. SPINORS

In Appendix A an irreducible set of five 4 x 4 complex matrix valued functions of points of V_5, $\gamma_\alpha(x)$ is constructed and their algebraic properties are discussed. These $\gamma_\alpha(x)$ satisfy

(3.1) $$\gamma_\alpha(x)\gamma_\beta(x) + \gamma_\beta(x)\gamma_\alpha(x) = 2\gamma_{\alpha\beta}(x)\,1_4 \qquad (\alpha, \beta = 1, 2, 3, 4, 5)$$

in every coordinate system of V_5. Thus under a coordinate transformation in V_5 of the form

$$x^{*\alpha} = x^{*\alpha}(x)$$

we have

(3.2) $$\gamma^*_\alpha(x^*) = \gamma_\beta(x(x^*))\,\frac{\partial x^\beta}{\partial x^{*\alpha}}$$

that is the γ_α transform as covariant vector in V_5.

However, the γ_α are not uniquely determined by equations (3.1) for if $\gamma_\alpha(x)$ satisfy them as so do

$$\gamma'_\alpha = T\gamma_\alpha t \tag{3.3}$$

when $T(x)$ is a complex non-singular 4×4 matrix and $t(x)$ is its inverse, that is

$$tT = Tt = 1_4 \tag{3.4}$$

We regard T as a coordinate transformation in a four dimensional complex vector space $S_4(x)$ the spin space at the point x^α of V_5. The totality of $S_4(x)$ for x^α ranging over all points of V_5 is the spin bundle. Sections of this bundle, $\psi(x)$ are called spinors. Under the transformation $T(x)$ given by equations (3.3) ψ transforms as

$$\psi'(x) = T\psi(x)t^w \tag{3.5}$$

and is said to be a spinor of weight w.

The discussion of the covariant derivative of such spinors given below follows that given by Veblen and Taub [8]. We denote the covariant derivative of $\psi = \| \psi^L \|$ by

$$\psi_{;\alpha} = \psi_{,\alpha} + \Gamma_\alpha \psi - w \operatorname{trace}(\Gamma_\alpha)\psi \tag{3.6}$$

where

$$\Gamma_\alpha = \| \Gamma^L_{M\alpha} \|$$

is the spin connection and it has the transformation law

$$\Gamma'_\alpha = T(\Gamma_\alpha t + t_{,\beta}) \frac{\partial \bar{x}^\beta}{\partial x^\alpha}$$

under coordinate transformation in S_4 and V_5.

We shall require that the operation of taking the complex conjugate of spinors commute with the operation of taking their covariant derivative. We shall further require that

$$\gamma_{\alpha;\beta} = 0. \tag{3.7}$$

It then follows from the results given in the Appendix, equations (A.23) to (A.28) and (A.30), that

$$\gamma_{;\alpha} = 0. \tag{3.8}$$

It may also be shown as a consequence of equations (3.7) and the reality of the

Christoffel symbols of V_5 that

$$H_{;\beta} = 0 \tag{3.9}$$

where $H = \| H_{LM} \|$ is defined by the equations (A.32).

We assign to contravariant spinors ψ^L the weight 1/4, to γ the weight $-1/2$, to γ_α the weight zero, and to H the weight $-1/4$ and anti-weight $-1/4$. Then equations (3.7), (3.8) and (3.9) read

$$\gamma_{\alpha;\beta} = \gamma_{\alpha,\beta} - \gamma_\delta \Gamma^\delta_{\alpha\beta} + K_\beta \gamma_\alpha - \gamma_\alpha K_\beta = 0 \tag{3.10}$$

$$\gamma_{;\beta} = \gamma_{,\beta} - \gamma K_\beta + (\gamma K_\beta)^T = 0 \tag{3.11}$$

$$H_{;\beta} = H_{,\beta} - HK_\beta + (\overline{HK}_\beta)^T = 0 \tag{3.12}$$

where

$$K_\beta = \| K^L{}_{M\beta} \| = \Gamma_\beta - \frac{1}{4}\,(\text{trace } \Gamma_\beta)\, 1_4 \tag{3.13}$$

and the bar denotes the complex conjugate.

Multiplying equation (3.10) by γ^γ and taking the trace we find that

$$4\Gamma^\gamma_{\alpha\beta} = \text{trace}\,(2S_\alpha{}^\gamma K_\beta + \gamma^\gamma \gamma_{\alpha,\beta}). \tag{3.14}$$

These equations may be solved for the K_β by multiplying them by $\gamma_\gamma \gamma^\alpha$ and summing on γ and α, using equations (A.30), (A.31) and equations (3.11). One obtains

$$4K_\beta = \gamma_\alpha(\gamma^\alpha_{,\beta} + \gamma^\gamma \Gamma^\alpha_{\gamma\beta}) + 2\eta_\beta \tag{3.15}$$

where

$$\eta_\beta = \| \gamma^{RL} \gamma_{RM,\beta} \|. \tag{3.16}$$

When the matrices γ_α and γ^α are defined as in the appendix then the matrices γ_{LM} and γ^{LM} are constant matrices as is the matrix H_{LM}. Since η_β vanishes in this case, equation (3.15) may be written as

$$4K_\beta = S_{\underline{\alpha}}{}^{\underline{\delta}}\, \omega^{\underline{\alpha}}{}_{\underline{\delta}\beta} \tag{3.17}$$

where $\omega^{\underline{\alpha}}{}_{\underline{\delta}\beta}$ is given by equations (2.12). Note that equations (3.10) and (3.11) then state that

$$(\gamma K_\beta)^T = \gamma K_\beta$$

that is, γK_β is symmetric and that

$$(\overline{HK}_\beta)^T = HK_\beta$$

that is, HK_β is hermitian.

Equations (3.17) may be written as

$$4K_5 = W\left[F_{ij}\tilde{S}^{ij} + 2\,\frac{W_{,i}}{W}\,\gamma_{\underline{5}}\tilde{\gamma}^i\right] \tag{3.18}$$

$$4K_i = 4\tilde{K}_i + 4A_iK_5 - 2F_{ij}\tilde{\gamma}^j\gamma_{\underline{5}} \tag{3.19}$$

where $\tilde{K}_i$ and $\tilde{\gamma}^i$ are the spin connection and γ matrices respectively determined by the metric tensor g_{ij}. These equations hold when the coordinate system in each fibre of the spin bundle is such that the matrices $\gamma_\alpha(x)$ are given by equations (A.38). Equations (3.15) may be used to calculate K_β when the coordinate systems in the fibres of the spin bundle vary from point to point in V_5. That is when the matrices $\gamma_\alpha(x)$ are replaced by

$$\gamma_\alpha' = T(x)\gamma_\alpha t(x). \tag{3.25}$$

The matrices $\tilde{\gamma}^i$, $\tilde{\gamma}_i$, $\tilde{S}_{ij}$, and $\tilde{S}^{ij}$ are given in terms of γ^i and γ_i in the appendix.

4. THE SPIN CURVATURE

It is a consequence of equations (3.6) that for spinors of weight 1/4

$$\psi_{;\alpha\beta} - \psi_{;\beta\alpha} = -\,\mathcal{B}_{\alpha\beta}\psi$$

where the matrix $\mathcal{B}_{\alpha\beta} = \| \mathcal{B}^L{}_{M\alpha\beta}\|$ is given by

$$\mathcal{B}_{\alpha\beta} = K_{\beta,\alpha} - K_{\alpha,\beta} + K_\alpha K_\beta - K_\beta K_\alpha\,. \tag{4.1}$$

The rules of covariant differentiation enable us to write

$$\gamma_{\alpha;\beta\gamma} - \gamma_{\alpha;\gamma\beta} = \gamma_\delta B^\delta{}_{\alpha\beta\gamma} + \gamma_\alpha\mathcal{B}_{\beta\gamma} - \mathcal{B}_{\beta\gamma}\gamma_\alpha = 0. \tag{4.2}$$

These equations may be solved for $B_{\epsilon\alpha\beta\gamma}$ by multiplying these by γ_ϵ and taking the trace to obtain

$$2B_{\epsilon\alpha\beta\gamma} = -\,\mathrm{trace}\,(S_{\epsilon\alpha}\mathcal{B}_{\beta\gamma}). \tag{4.3}$$

It may be verified from equation (A.18) that

$$4\,\mathcal{B}_{\beta\gamma} = B_{\beta\gamma\epsilon\delta}S^{\epsilon\delta} \tag{4.4}$$

satisfies equation (6.3). Also since trace $K_\beta = 0$ it follows that

$$\mathrm{trace}\;\mathcal{B}_{\beta\gamma} = 0. \tag{4.5}$$

It is a consequence of equations (4.1) and (3.11) that

(4.6) $$(\gamma B_{\alpha\beta})^T = \gamma B_{\alpha\beta}.$$

This equation also follows from equation (4.5).

5. THE FIVE-DIMENSIONAL ENERGY MOMENTUM VECTOR

Since the field equations of the five dimensional theories we are concerned with are derived from a variational principle involving the scalar curvature of V_5 one may use the arguments of J.N. Goldberg [10] and J. Isenberg and J. Nestor [11] to show that the total energy momentum 5-vector for an asymptotically flat V_5 is given in terms of its Christoffel symbol by

(5.1) $$16\pi G_5 P_\alpha U^\alpha = \oint_S U^\lambda \delta^{\alpha\beta\sigma}_{\xi\upsilon\lambda} g^{\nu\delta} \Delta\Gamma^\mu_{\delta\beta} \frac{1}{2} d\sigma_{\sigma\alpha}$$

where U^α is a constant vector, G_5 is a constant we shall relate to Newton's gravitational constant G,

$$\Delta\Gamma^\mu_{\delta\beta} = \Gamma^\mu_{\delta\beta} - {}_0\Gamma^\mu_{\delta\beta} = 0\left(\frac{1}{r^2}\right) \quad ,$$

$\Gamma^\mu_{\delta\beta}$ and ${}_0\Gamma^\mu_{\delta\beta}$ are the Chirstoffel symbols computed from the metric of V_5 and the flat metric to which it approaches respectively, is a three surface at infinity and $d\sigma_{\sigma\alpha}$ is its area two form.

It may be verified that when

(5.2) $$\gamma_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + 0\left(\frac{1}{r^2}\right)$$

then

(5.3) $$U^\lambda \delta^{\alpha\beta\sigma}_{\mu\nu\lambda} g^{\nu\delta} \Delta\Gamma^\mu_{\delta\beta} = U^\lambda \delta^{\alpha\beta\sigma}_{\mu\nu\lambda} \eta^{\nu\delta} \eta^{\mu\rho}(h_{\beta\rho,\delta} - h_{\beta\delta,\rho})$$

In equations (5.1) to (5.3) inclusive the indices range from 1 to 5. If they are restricted to range from 1 to 4 and G_5 is replaced by G, these equations reduce to those that obtain in a four-dimensional space-time.

Following the observation of Nester [3] and Moreschi and Sparling [5] we note that we may write equation (5.1) as

(5.4) $$16\Pi G_5 P_\alpha U^\alpha = \oint_S E^{\alpha\beta} \frac{1}{2} d\sigma_{\alpha\beta}$$

with

(5.5) $$E^{\alpha\beta} = E^{\alpha\beta\gamma\delta\epsilon}(\psi^* S_{\gamma\delta}\,\psi_{;\epsilon} - \psi^*_{;\epsilon} S_{\gamma\delta}\,\psi)$$

where the quantities entering into this equation are defined as follows:

(5.6) $$E^{\alpha\beta\gamma\delta\epsilon} = (\gamma)^{-1/2}\epsilon^{\alpha\beta\gamma\delta\epsilon}$$

is a pure imaginary quantity with $\epsilon^{\alpha\beta\gamma\delta\epsilon}$ the Levi-Civita alternating tensor density,

(5.7) $$\psi^* = \bar{\psi}^T H,$$

as above the semi-colon denotes the covariant derivative and the $S_{\alpha\beta}$ are defined by the analogues of equations (A.14), namely

(5.8) $$2S_{\alpha\beta} = \gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha .$$

When the $\gamma_{\alpha\beta}$ are given by equations (5.2) we have

$$e^{\gamma}_{\underline{\alpha}} = \delta^{\gamma}_{\underline{\alpha}} - \frac{1}{2}\, h_{\alpha\beta}\eta^{\beta\gamma} + 0\left(\frac{1}{r^2}\right)$$

(5.9) $$\omega^{\underline{\alpha}}_{\gamma} = \delta^{\underline{\alpha}}_{\gamma} + \frac{1}{2}\, h_{\gamma\beta}\eta^{\beta\alpha} + 0\left(\frac{1}{r^2}\right)$$

(5.10) $$2\omega_{\underline{\alpha}\underline{\beta}\epsilon} = h_{\epsilon\alpha,\beta} - h_{\epsilon\beta,\alpha} .$$

It then follows from equations (3.17) and (A.15), the reality of $\psi^* S_{\alpha\beta}\psi$ and the fact that

$$\omega_{\underline{\alpha}\underline{\beta}\underline{\epsilon}} + \omega_{\underline{\beta}\underline{\epsilon}\underline{\alpha}} + \omega_{\underline{\epsilon}\underline{\alpha}\underline{\beta}} = 0$$

that

$$\psi^* S_{\underline{\gamma}\underline{\delta}}\,\psi_{;\epsilon} - \psi^*_{;\epsilon} S_{\underline{\gamma}\underline{\delta}}\,\psi = \frac{1}{2}\, E_{\underline{\gamma}\underline{\delta}\underline{\lambda}\underline{\mu}\underline{\nu}} U^{\underline{\nu}}\omega^{\underline{\lambda}\underline{\mu}}_{\epsilon}$$

where

(5.11) $$U^{\alpha} = \psi^*_0\gamma^{\alpha}\psi_0 .$$

Hence

$$E^{\alpha\beta} = \delta^{\alpha\beta\epsilon}_{\underline{\lambda}\underline{\mu}\underline{\nu}}\, U^{\underline{\nu}}\omega^{\underline{\lambda}\underline{\mu}}_{\epsilon}$$

and is equal to the integrand in euqation (5.1).

Equation (5.1) may then be written as

(5.12) $$16\pi G_5 P_\alpha U^\alpha = \oint_S \frac{1}{2} E^{\alpha\beta} \mathrm{d}\sigma_{\alpha\beta} = \int_\Sigma E^{\alpha\beta}{}_{;\beta} n_\alpha \mathrm{d}_4 v$$

where Σ is a four dimensional hypersurface bounded by S, n_α is it unit normal, $\mathrm{d}_4 v$ is its invariant volume element, and Stokes' theorem has been applied.

We now turn to the evaluation of the four dimensional integral in equations (5.12). It follows from equations (5.5), (4.4), (A.16) and (A.17) that

(5.13) $$E^{\alpha\beta}{}_{;\beta} = -4(\psi^*_{;\beta}\gamma^\beta \psi_{;\epsilon}\gamma^{\epsilon\alpha} + \psi^*_{;\beta}\gamma^\epsilon \psi_{;\epsilon}\gamma^{\beta\alpha} - \psi^*_{;\beta}\gamma^\alpha \psi_{;\epsilon}\gamma^{\beta\epsilon} - - \psi^*_{;\beta}\gamma^\beta\gamma^\alpha\gamma^\epsilon \psi_{;\epsilon}) + 2U^\beta \Gamma^\alpha{}_\beta$$

where $\Gamma^\alpha{}_\beta$ is the Einstein tensor formed from $\gamma_{\alpha\beta}$, (it has been evaluated for the case where equation (1.1) holds in section 2),

$$U^\beta = \psi^* \gamma^\beta \psi$$

and is a future oriented non-space-like vector.

We shall assume that the hypersurface Σ is space-like and has as its normal the future pointing time-like vector

$$n^\alpha = e^\alpha_{\underline{4}}$$

that is

$$n_\alpha = -\omega^{\underline{4}}_\alpha$$

Then

(5.14) $$n_\alpha E^{\alpha\beta}_{;\beta} = -\omega^{\underline{4}}_\alpha E^{\alpha\beta}_{;\beta} = +2\Gamma_{\alpha\beta} e_{\underline{4}}{}^\alpha U^\beta - 4(\psi^*_{;\beta}\gamma^{\underline{4}} \psi_{;\epsilon} k^{\beta\epsilon} + \psi^*_{;\beta}\hat{\gamma}^\beta\gamma^{\underline{4}}\hat{\gamma}^\epsilon \psi_{;\epsilon})$$

where

(5.15) $$\gamma^{\underline{4}} = -n_\alpha \gamma^\alpha, \qquad \hat{\gamma}_\beta = k^{\beta\epsilon}\gamma_\epsilon = (\gamma^{\beta\epsilon} + e^\beta_{\underline{4}} e^\epsilon_{\underline{4}})\gamma_\epsilon,$$

and thus $k^{\beta\epsilon}$ is the induced positive definite metric on Σ.

It then follows that

(5.16) $$16\pi G_5 P_\alpha U^\alpha = \oint_S E^{\alpha\beta} \frac{\mathrm{d}\sigma_{\alpha\beta}}{2} \leqslant 0$$

under the assumptions that the analogue of the Witten equation, the equation

(5.17) $$\hat{\gamma}^\epsilon \psi_{;\epsilon} = 0$$

holds, and that

(5.18) $$\Gamma_{\alpha\beta} U^\alpha n^\beta \leqslant 0$$

for arbitrary future pointing non-spacelike vectors U^α and n^α. Equation (5.17) will be discussed in Section 7.

The inequality (5.16) implies that for any Lorentzian five dimensional asymptotically flat space V_5 for which equations (5.17) and the inequality (5.18) hold has positive energy, that is P^α is a future pointing non spacelike vector.

When V_5 admits a killing vector ζ^α, n^α in the inequality (5.18) is restricted to be such that

$$n_\alpha \zeta^\alpha = 0,$$

$$W = 1$$

and

$$G_{\underline{ab}} = -8\pi G(T^M_{\underline{ab}} + T^E_{\underline{ab}}),$$

the inequality (5.18) implies that

$$2T^M_{\underline{ab}} V^{\underline{a}} n^{\underline{b}} \geqslant (4\pi)^{1/2} G^{-1/2} |J_{\underline{a}} n^{\underline{a}}|$$

where

$$V^{\underline{a}} = \frac{U^{\underline{a}}}{|U^{\underline{5}}|}$$

is an arbitrary future pointing non-space-like vector in V_4 as is $n^{\underline{a}}$.

6. THE INTEGRAL OVER S

In evaluating this integral for the case where equation (1.1) holds, we shall assume that the three dimensional space S consists of the direct product of S^1, a circle over which x^5 varies, and S' a two dimensional surface in Σ', the three dimensional space x^4 = constant and x^5 = constant. We denote by r the distance of points in Σ' from a fixed point (the origin and take as S' the points of Σ' for which r takes on a large constant value.

We also assume that V_5 has a metric tensor given by equation (1.1) and that on S

$$0 < x^5 \leqslant 2\pi R_0$$

$$\psi = \psi_0 + 0\left(\frac{1}{r}\right)$$

$$W = 1 + 0\left(\frac{1}{r}\right)$$

$$A_i = 0\left(\frac{1}{r}\right)$$

and neglect terms of order r^{-n} with $n > 2$.

We shall further assume that the three surface S is given by the parametric equations

$$\begin{aligned} & x^i = x^i(u^1, u^2) \qquad (i \neq 4, 5) \\ & x^4 = \text{constant} \\ & x^5 = u^3. \end{aligned} \tag{6.1}$$

Then

$$\mathrm{d}\sigma_{\alpha\beta} = \frac{i}{3!} E_{\alpha\beta\gamma\delta\epsilon}\mathrm{d}\tau^{\gamma\delta\epsilon} \tag{6.2}$$

where

$$\mathrm{d}\tau^{\gamma\delta\epsilon} = \epsilon^{abc} \frac{\partial x^\gamma}{\partial u^a} \frac{\partial x^\delta}{\partial u^b} \frac{\partial x^\epsilon}{\partial u^c} \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3, \qquad (a, b = 1, 2, 3). \tag{6.3}$$

It follows from equations (6.1) that

$$2\mathrm{d}\tau^{\gamma\delta\epsilon} = (\delta_5^\gamma \delta_{ij}^{\delta\epsilon} + \delta_5^\delta \delta_{ij}^{\epsilon\gamma} + \delta_5^\epsilon \delta_{ij}^{\gamma\delta})\, \mathrm{d}\tau^{ij}\, \mathrm{d}x^5$$

where

$$\begin{aligned} \mathrm{d}\tau^{ij} &= \left(\frac{\partial x^i}{\partial u^1} \frac{\partial x^j}{\partial u^2} - \frac{\partial x^i}{\partial u^2} \frac{\partial x^j}{\partial u^1}\right) \mathrm{d}u^1 \mathrm{d}u^2 = \\ &= -\frac{i}{2} E^{ijkl} \mathrm{d}\tilde{\sigma}_{\kappa l}. \end{aligned} \tag{6.4}$$

It then follows from equations (5.5) that

$$\begin{aligned} \frac{1}{2} E^{\alpha\beta} \mathrm{d}\sigma_{\alpha\beta} = 2i\Big[& \psi^*\gamma_5 \tilde{\gamma}_i \psi_{;j} - \psi^*_{;j}\gamma_5 \tilde{\gamma}_i \psi + \\ & + \frac{1}{2} (\psi^* S_{ij} \psi_{;5} - \psi^*_{;5} S_{ij} \psi)\Big] \mathrm{d}\tau^{ij} \mathrm{d}x^5 \end{aligned} \tag{6.5}$$

where

$$\tilde{\gamma}_i = \gamma_i - A_i\gamma_5 = \omega_i^a \gamma_a \tag{6.6}$$

and hence

$$\tilde{\gamma}_i\tilde{\gamma}_j + \tilde{\gamma}_j\tilde{\gamma}_i = 2g_{ij}1_4 . \tag{6.7}$$

From equations (3.18) and (3.19), and since ψ is assumed to be independent of x^5 we have

$$\begin{aligned} \psi_{;j} &= \psi_{|j} + A_j K_5 \psi - \frac{1}{2} F_{jk}\tilde{\gamma}^k\gamma_5\psi \\ \psi_{;5} &= K_5\psi = \frac{W}{4}\left[F_{ij}\tilde{S}^{ij} + 2\,\frac{W_{,i}}{W}\,\gamma_{\underline{5}}\tilde{\gamma}^i\right] \end{aligned} \tag{6.8}$$

with

$$\tilde{\gamma}^i = g^{ij}\tilde{\gamma}_j \tag{6.9}$$

$$2\tilde{S}^{ij} = \tilde{\gamma}^i\tilde{\gamma}^j - \tilde{\gamma}^j\tilde{\gamma}^i , \tag{6.10}$$

and the stroke denotes the covariant derivative of spinors over V_4.

It follows from equations (A.15), (A.18) and (6.8) that

$$\frac{1}{2}\, E^{\alpha\beta}\mathrm{d}\,\sigma_{\alpha\beta} = W\left[\tilde{E}^{ij}\,\frac{\mathrm{d}\,\tilde{\sigma}_{ij}}{2} + \left(iF^*_{ij}U_{\underline{5}} + iF_{ij}\psi^*\psi + {}\right.\right.$$

$$\left.\left. + iE_{ijkl}\tilde{U}^l g^{km}\,\frac{W_{,m}}{W}\right)\mathrm{d}\tau^{ij}\right]\mathrm{d}x^5$$

where

$$2F^*_{ij} = E_{ijkl}F^{kl},$$

a pure imaginary quantity as is $\psi^*\psi$,

$$U_{\underline{5}} = \psi^*\gamma_{\underline{5}}\psi$$

and

$$\tilde{U}^l = \psi^*\tilde{\gamma}^l\psi .$$

We define

$$i \oint_{S'} f_{ij}^* \, \mathrm{d}\tau^{ij} = 8\pi Q_E,$$

$$\oint_{S'} f_{ij} \, \mathrm{d}\tau^{ij} = 8\pi Q_M \tag{6.11}$$

$$i \oint_{S'} E_{ijkl} g^{km} W_{|m} \tilde{U}^l \mathrm{d}\tau^{ij} = 16\pi G w_n \tilde{U}^n .$$

The quantity Q_M vanishes when equation (2.20) holds throughout the two surface S', that is, when ϕ_i is non-singular. The vector w_n vanishes when W is constant.

In view of these definitions and the fact that for the case we are considering all quantities are independent of x^5, we have

$$\frac{1}{2\pi R_0} \oint \frac{1}{2} E^{\alpha\beta} \mathrm{d}\sigma_{\alpha\beta} = 16\pi G[(p_i + w_i)\tilde{U}^i + \\ + G^{-1/2}\pi^{1/2} Q_E U_{\underline{5}} + iG^{-1/2}\pi^{1/2} Q_M \psi^* \psi] \leqslant 0. \tag{6.12}$$

The inequality (6.12) when applied to the case where $\tilde{U}^i = \delta_4^i$ implies that

$$m = -p_4 \geqslant w_4 + G^{-1/2}\pi^{1/2}(Q_E^2 + Q_n^2)^{1/2} . \tag{6.13}$$

Aside from the units used, this result is that given by Moreschi and Sparling [5] and differs from that of Gibbons and Hull [6] by a factor of 1/2 in the charges.

If for arbitrary hypersurfaces t = constant, equation (5.17) holds and the inequalities (6.12) and (6.13) are replaced by equalities we must have

$$\Gamma_{\alpha\beta} = 0 \tag{6.14}$$

$$k^{\beta\epsilon}\psi_{;\epsilon} = 0 \tag{6.15}$$

for each such hypersurfaces. That is ψ must be a covariantly constant spinor field in V_5, in other words, we must have

$$\psi_{;\epsilon} = 0. \tag{6.16}$$

We shall prove in section 8 that if a V_5 with metric given by (1.1) is regular, equations (6.14) hold, V_5 is asymptotically flat and admits solutions of equation (6.15) independent of x^5, then it has a vanishing curvature tensor that is, is flat.

The proof consists of showing tha such V_5's are stationary and hence Lichnerowicz's results on stationary Klein-Kaluza and Jordan-Thiry theories [1] apply. Thus we find that spaces V_5 satisfying the above conditions which have vanishing energy-momentum vectors P_α are flat.

7. THE WITTEN EQUATION IN V_5

This equation is given by equation (5.17), namely

$$\hat{\gamma}^{\beta}\psi_{;\beta} = 0 \tag{7.1}$$

where $\hat{\gamma}^{\beta}$ is defined in terms of γ_α by equation (5.15). We want to discuss solutions of this equation that are independent of x^5. We shall first reduce equation (7.1) to an equation in V_4, the four dimensional space with metric tensor g_{ij}. We observe that it follows from equation (5.15) that

$$\hat{\gamma}^{i} = k^{i\alpha}\gamma_\alpha = h^{ij}(\gamma_j - A_j\gamma_5) = h^{ij}\tilde{\gamma}_j \tag{7.2}$$

where

$$h^{ij} = g^{ij} + e^{i}_{\underline{4}}e^{j}_{\underline{4}}. \tag{7.3}$$

The tensor h_{ij} is the metric induced by the metric g_{ij} on the subspace $t = x^4 =$ $=$ constant in V_4.

We also have

$$\hat{\gamma}^{5} = W^{-1}\gamma_5 - h^{ij}\tilde{\gamma}_i A_j. \tag{7.4}$$

It is a consequence of equations (7.2), (7.4) and (3.24) that equation (7.1) may be written as

$$h^{ij}\tilde{\gamma}_j\psi_{|i} = A\psi \tag{7.5}$$

where

$$A = \frac{1}{2}\,h^{ij}\tilde{\gamma}_j\tilde{\gamma}^k F_{ik} - W^{-1}\gamma_{\underline{5}}K_5$$

and K_5 is given by equation (3.23). Thus we have

$$A = \frac{1}{4}\left[F_{ij}\tilde{S}^{ij}\gamma_{\underline{5}} + 2F_{\underline{4}k}\tilde{\gamma}_4\tilde{\gamma}^k\gamma_{\underline{5}} - 2\,\frac{W_{,i}}{W}\,\tilde{\gamma}^i\right]. \tag{7.6}$$

The existence of solutions of equations (7.5) with general spin matrices A has been proven by T.H. Parker [7].

8. COVARIANT CONSTANT SPINORS IN V_5

Such spinors satisfy the equation

$$\psi_{;\alpha} = 0 \tag{8.1}$$

If in addition they are independent of x^5, that is if

$$\psi_{,5} = K_5 \psi = 0 \tag{8.2}$$

and equations (8.1) and (8.2) may be written as

$$\psi_{;j} = \psi_{|i} - \frac{1}{2} F_{ij} \tilde{\gamma}^j \gamma_5 \psi = 0 \tag{8.3}$$

and

$$\left[F_{ij} \tilde{S}^{ij} + 2 \frac{W_{,i}}{W} \gamma_5 \tilde{\gamma}^i \right] \psi = 0 \tag{8.4}$$

respectively. The first of equations (8.3) are similar to but different from the equations used by Gibbons and Hull in their introduction of a supercovariant derivative of spinors [6].

The vector field ξ^α defined as

$$\xi^\alpha = \psi^* \gamma^\alpha \psi \tag{8.5}$$

where ψ is covariantly constant, that is, satisfies equations (8.3) and (8.4) may be shown to be timelike except when

$$F_{ij} = W_{,k} = 0. \tag{8.6}$$

It is a consequence of equations (8.1) and (8.2) that

$$\xi^\alpha_{;\beta} = \xi^\alpha_{,5} = 0. \tag{8.7}$$

Hence ξ^α is a time-like Killing vector in V_5. That is if equations (8.1) hold V_5 is stationary and when equations (1.1) hold, the Killing vector ξ^α is independent of x^5.

We next show that the space-time V_4 with metric g_{ij} is stationary that is, admits a time-like Killing vector $\tilde{\xi}^i$ such that the Lie derivative of W and F_{ij} with respect to $\tilde{\xi}^i$ vanishes. The Killing equations in V_5 are

$$\xi^\gamma \gamma_{\alpha\beta,\gamma} + \gamma_{\alpha\gamma} \xi^\gamma_{,\beta} + \gamma_{\gamma\beta} \xi^\alpha_{,\beta} = 0. \tag{8.8}$$

By setting $\alpha = \beta = 5$ one may show that

$$\xi^i W_{,i} = \tilde{\xi}^i W_{,i} = 0 \tag{8.9}$$

where

(8.10) $$\tilde{\xi}^i = \psi^* \tilde{\gamma}^i \psi = g^{ij} \tilde{\xi}_j.$$

On setting $\alpha = 5, \beta = i$ in equations (8.8) one finds that

(8.11) $$\tilde{\xi}^i A_{i|j} + A_j \tilde{\xi}^j_{|i} + \xi^5{}_{|i} = 0.$$

It is a consequence of the relation between γ^5 and γ_5 that

(8.12) $$\xi^5 = W^{-1} \xi_{\underline{5}} - A_j \tilde{\xi}^j.$$

On substituting this equation into equation (8.11) one finds that

(8.13) $$\tilde{\xi}^j F_{ij} = -\frac{W}{2} (W^{-1} \xi_{\underline{5}})_{,i}.$$

Since

$$\xi_{\underline{5}} = \psi^* \gamma_{\underline{5}} \psi$$

it follows from equations (8.3) that

$$\xi_{\underline{5}|i} = -F_{ij} \tilde{\xi}^j.$$

It is a consequence of this result and equation (8.13) that

(8.14) $$(\xi_5)_{,i} = 0,$$

that is, ξ_5 is constant.

Equations (8.3) and (8.10) imply that

(8.15) $$\tilde{\xi}_{j|i} = F_{ij} \xi_{\underline{5}}$$

and hence

(8.16) $$\tilde{\xi}_{i|j} + \tilde{\xi}_{i|j} = 0$$

that is $\tilde{\xi}^i$ is a time-like Killing vector of the space-time V_4 with metric tensor g_{ij}. It follows from equations (8.9) and (8.11) that the Lie derivative with respect to $\tilde{\xi}^i$ of the tensor F_{ij} vanishes, that is

$$\tilde{\xi}^k F_{ij,k} + F_{ik} \tilde{\xi}^k_{,j} + F_{kj} \tilde{\xi}^k_{,i} = 0.$$

On making the transformation of coordinates in V_5 of the form

$$x^{5*} = x^5 + \psi(x^i)$$

$$x^{*i} = \phi^i(x^j)$$

and thereby insuring that

(8.17) $$\xi^{\alpha *} = \delta^{\alpha}_{4},$$

while

$$\zeta^{\alpha} = \delta^{\alpha}_{5}$$

is also a Killing vector in V_5. In such a coordinate system equation (8.11) becomes

$$\tilde{\xi}^{j} A_{i|j} + A_j \tilde{\xi}^{j}_{,i} = 0.$$

Thus V_4 admits a one parameter group of isometries generated by the Killing vector $\tilde{\xi}^{i}$, under which the scalar W and the vector field A_i are invariant. If the hypersurfaces x^5 = constant in V_5 are globally isomorphic to the direct product of a three space V_3 and the real line, then V_5 is isomorphic to the product $V_3 \times \times R \times S^1$ and is said to be a stationary space. It is said to be complete if V_3 is complete and asymptotically flat if V_3 is.

Lichnerowicz has shown [1] that enerywhere regular, asympotically flat (or complete) stationary spaces V_5 with metrics given by equation (1.1) for which $\Gamma_{\alpha\beta} = 0$ are flat, that is have

$$B^{\alpha}{}_{\beta\gamma\delta} = 0$$

everywhere. Thus the spaces V_5 which are everywhere regular, asymptotically flat, have Einstein tensors satisfying the inequality (5.18) and have vanishing energy-momentum vectors P_{α} are flat.

9. THE SPACE $\overline{V}_5$

This space is a five dimensional space whose metric tensor $\overline{\gamma}_{\alpha\beta}$ is conformal to that of V_5, that is

(9.1) $$\overline{\gamma}_{\alpha\beta} = e^{2\sigma} \gamma_{\alpha\beta}$$

where $\gamma_{\alpha\beta}$ is given by equation (1.1). We again define

(9.2) $$\overline{g}_{\mu\nu} = \gamma_{\mu\nu} - \frac{\gamma_{5\mu}\gamma_{5\nu}}{\gamma_{55}} = e^{2\sigma} g_{\mu\nu}$$

and have

(9.3) $$\overline{\gamma}_{55} = \overline{W}^2 = e^{2\sigma} W^2 = e^{2\sigma} \gamma_{55}$$

(9.4) $$(\overline{\gamma}_{55})^{-1} \overline{\gamma}_{5\mu} = \overline{A}_{\mu} = A_{\mu} = (\gamma_{55})^{-1} \gamma_{5\mu}.$$

The function $\sigma(x^i)$ is determined by the requirement that the integral

$$\bar{I} = \int \sqrt{-\bar{\gamma}}\,\bar{B}\,\mathrm{d}^5 x,$$

where $\bar{\gamma}$ is the determinant of $\bar{\gamma}_{\alpha\beta}$, and $\bar{B}$ is the scalar curvature of $\bar{V}_5$, such that the coefficient of $\sqrt{-\bar{g}}\,\bar{R}$ is constant when g is the determinant of $\bar{g}_{\mu\nu}$ and $\bar{R}$ is the scalar curvature computed from $\bar{g}_{\mu\nu}$. This condition implies that

$$e^{\sigma} = W^{-1/3}.$$

It then follows that

$$\bar{\Gamma}_{\alpha\beta} = \bar{B}_{\alpha\beta} - \frac{1}{2}\,\bar{\gamma}_{\alpha\beta}\bar{B},$$

where $\bar{B}_{\alpha\beta}$ and $\bar{B}$ are the Ricci tensor and scalar curvature determined from $\bar{\gamma}_{\alpha\beta}$, are given by

$$(9.5)\qquad \bar{\Gamma}_{\alpha\beta} = \Gamma_{\alpha\beta} - \left[\frac{W_{;\alpha\beta}}{W} - \frac{2}{3}\,\frac{W_{;\alpha}W_{;\beta}}{W^2}\right] + \gamma_{\alpha\beta}\left[\frac{W_{;\gamma\delta}}{W} - \frac{2}{3}\,\frac{W_{;\gamma}W_{;\delta}}{W^2}\right]\gamma^{\gamma\delta}.$$

In these equations the semi-colon denotes the covariant derivative with respect to $\gamma_{\alpha\beta}$.

It may be shown that

$$\bar{\Gamma}_{\underline{a}\underline{b}} = \bar{\Gamma}_{\alpha\beta}\bar{e}^{\alpha}_{\underline{a}}\bar{e}^{\beta}_{\underline{b}} = e^{-2\sigma}\bar{\Gamma}_{\alpha\beta}e^{\alpha}_{\underline{a}}e^{\beta}_{\underline{b}}$$

is given by

$$(9.6)\qquad e^{2\sigma}\bar{\Gamma}_{\underline{a}\underline{b}} = G_{\underline{a}\underline{b}} + 8\pi G W^2 T^E_{\underline{a}\underline{b}} + T^F_{\underline{a}\underline{b}}$$

where T^E_{ab} is given by equation (2.21) and

$$(9.7)\qquad 3W^2 T^F_{\underline{a}\underline{b}} = 2\left[W_{,\underline{a}}W_{,\underline{b}} - \frac{1}{2}\,\eta_{\underline{a}\underline{b}}\eta^{\underline{\gamma}\underline{\delta}}W_{,\underline{\gamma}}W_{,\underline{\delta}}\right].$$

It has been shown (cf. [13]) that T^F_{ab} is the stress energy tensor of a perfect fluid that satisfies the equation of state

$$p = w$$

where p is the pressure and w is the energy density; and further the motion is irrotational. That is,

$$T^F_{ab} = (w + p)\,\phi_{\underline{a}}\phi_{\underline{b}} - p\eta_{\underline{a}\underline{b}}$$

where

(9.8)
$$w = p = \frac{1}{3}\,\frac{W_{,i}W_{,j}g^{ij}}{W^2}\,,$$
$$\chi_i = \frac{W_{,i}}{\sqrt{W_{ij}W_{ik}g^{jk}}}\,.$$

The relation between the scalar χ and the scalar that occurs in the Brans-Dicke theory of gravitation is given in [13].

In addition we have

(9.9)
$$e^{2\sigma}\overline{\Gamma}_{\underline{5}\underline{b}} = \frac{1}{W}\,(WF_{\underline{b}}{}^{\underline{c}})_{|\underline{c}}$$

and

(9.10)
$$e^{2\sigma}\overline{\Gamma}_{\underline{5}\underline{5}} = -\frac{1}{2}\,(R + 3F^{\underline{a}\underline{b}}F_{\underline{a}\underline{b}}) + \frac{\Delta W}{W} - \frac{4}{3}\,\eta^{\underline{a}\underline{b}}\,\frac{W_{,\underline{a}}W_{,\underline{b}}}{W^2}$$

where $F_{\underline{a}\underline{b}}$ is given by equation (2.13).

The spin connection $\overline{K}_\gamma$ in $\overline{V}_5$ is related to K_γ of V_5 by the equations

(9.11)
$$\overline{K}_\gamma = K_\gamma + \frac{W_{,\delta}}{6W}\,S^\delta_\gamma\,.$$

It then follows from equations (3.18) and (3.19) that

(9.12)
$$4\overline{K}_5 = W\left[F_{ij}\widetilde{S}^{ij} + \frac{4W_{,i}}{3W}\,\gamma_{\underline{5}}\widetilde{\gamma}^i\right]$$
$$4\overline{K}_i = 4\widetilde{K}_i + 4A_i\overline{K}_5 - 2F_{ij}\widetilde{\gamma}^j\gamma_{\underline{5}} + \frac{2}{3}\,\frac{W_{,j}}{W}\,\widetilde{S}^j_i\,.$$

The discussion of §5 may be applied to the space $\overline{V}_5$ when the γ_α and the spin--covariant derivatives are replaced by

(9.13)
$$\overline{\gamma}_\alpha = e^\sigma\gamma_\alpha$$

and

(9.14)
$$\psi_{:\alpha} = \psi_{,\alpha} + \overline{K}_\alpha\,\psi$$

respectively. Then it follows that $\overline{P}^\alpha$, the total energy momentum 5-vector in $\overline{V}_5$, is a future pointing non space-like vector, that is, $\overline{V}_5$ has positive energy when

(9.15)
$$\hat{\overline{\gamma}}^\epsilon\,\psi_{:\epsilon} = e^{-\sigma}\hat{\gamma}^\epsilon\,\psi_{:\epsilon} = 0$$

and

$$\overline{\Gamma}_{\alpha\beta}\overline{U}^{\alpha}\overline{n}^{\beta} = e^{-2\sigma}\overline{\Gamma}_{\alpha\beta}U^{\alpha}n^{\beta} \leqslant 0 \tag{9.16}$$

where the colon denotes the covariant derivative with respect to the spin connection $\overline{K}_{\alpha}$ and $\overline{\Gamma}_{\alpha\beta}$ is given above.

It follows from equations (9.12), the discussion similar to that given in §7, and Parker's results [7], that solutions of equations (9.15) exist.

When V_5 admits a Killing vector ξ^{α} so does $\overline{V}_5$. If n^{α} is restricted to be such that $n_{\alpha}\xi^{\alpha} = 0$ and the coordinate system in $\overline{V}_5$ is such that $\xi^{\alpha} = \delta^{\alpha}_{5}$, the inequality (9.16) implies that in this coordinate system

$$\overline{U}^{\underline{a}}\overline{n}^{\underline{b}}(G_{\underline{ab}} + 8\pi W^2 T^E_{\underline{ab}} + T^F_{\underline{ab}}) + \frac{\overline{U}^5}{2W}\, n^{\underline{b}}(W^2(16\pi G)^{1/2} f_{\underline{b}}^{\underline{c}})_{|\underline{c}} \leqslant 0.$$

Setting

$$G_{\underline{ab}} = -8\pi G\left(T^M_{\underline{ab}} + W^2 T^E_{\underline{ab}} + \frac{1}{8\pi G}\, T^F_{\underline{ab}}\right) \tag{9.17}$$

we have

$$T^M_{\underline{ab}}\overline{V}^{\underline{a}}\overline{n}^{\underline{b}} \geqslant (4\pi)^{1/2} G^{-1/2}\left|\frac{W}{2} J_{\underline{b}} n^{\underline{b}} + \frac{1}{4\pi} W_{,\underline{c}} f_{\underline{b}}^{\underline{c}}\right| \tag{9.18}$$

where $T^E_{\underline{ab}}$ is given by equation (2.21), f_{ij} is defined by equation (2.20) and $J_{\underline{a}}$ is defined by the first of equations (2.25).

Equations (9.17) state that the source of the gravitational field g_{ij} depends linearly on the stress-energy tensor of the electromagnetic field described by ϕ_i, that of the scalar field W (which in turn describes the variation of the «gravitational constant») and the tensor T^M_{ab}, the stress-energy tensor of additional fields arising in particular problems.

Appendix

THE MATRICES γ_α, γ AND H

A. The Matrices γ_α, γ and H

In this appendix which, is based on the notes of a seminar on «Geometry of Complex Domains» conducted by O. Veblen and J. von Neumann in 1935-36 [12], we construct and discuss the algebraic properties of a set of $2v+1$, $2^v \times 2^v$ matrices β_α $(\alpha = 1, 2, 3, \ldots, 2v+1)$ which satisfy the equation

(A.1) $$\beta_\alpha \beta_\beta + \beta_\beta \beta_\alpha = 2\delta_{\alpha\beta} 1_{2^v}.$$

We shall subsequently set $v = 2$ and then define matrices γ_α which satisfy

(A.2) $$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\gamma_{\alpha\beta} 1_4$$

where $\gamma_{\alpha\beta}$ is the metric tensor of V_5. We begin with the three 2×2 matrices

(A.3) $$\beta_1^{(1)} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \beta_2^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \beta_0^{(1)} = -i\beta_1^{(1)}\beta_2^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and define

(A.4) $$\beta_p^{(v)} = \begin{pmatrix} \beta_p^{(v-1)} & 0 \\ 0 & -\beta_p^{(v-1)} \end{pmatrix} \quad p = 0, 1, \ldots, 2(v-1)$$
$$\beta_{2v-1}^{(v)} = \begin{pmatrix} 0 & i1_k \\ -i1_k & 0 \end{pmatrix} \qquad \beta_{2v}^{(v)} = \begin{pmatrix} 0 & 1_k \\ 1_k & 0 \end{pmatrix}$$

where 1_k is the unit $2^{v-1} \times 2^{v-1}$ matrix.

It follows by induction that

(A.5) $$\beta_0^{(v)} = (-i)^v \beta_1^{(v)} \ldots \beta_{2v}^{(v)}$$

and that (on omitting the superscript v)

(A.6) $$\beta_\alpha^T = (-1)^\alpha \beta_\alpha; \quad \overline{\beta}_\alpha = (-1)^\alpha \beta_\alpha; \quad \overline{\beta}_\alpha^T = \beta_\alpha$$

where A^T is the transpose of the matrix A and α is not summed in these equations. That is, the matrices β_α are hermitian. The matrices

$$1, \beta_\alpha, \beta_\alpha\beta_\beta(\alpha < \beta), \beta_\alpha\beta_\beta\beta_\gamma(\alpha < \beta < \gamma) \ldots$$
$$\beta_{\alpha_1}\beta_{\alpha_2} \ldots \beta_{\alpha_v}(\alpha_1 < \alpha_2 < \ldots < \alpha_v)$$

are a set of 2^{υ} linearly independent matrices which form a basis for all $2^{\upsilon} \times 2^{\upsilon}$ complex matrices.

We next define

$$
\begin{aligned}
&\gamma_{\underline{i}} = \beta_i (i = 1, 2, 3, 5, \ldots, 2\upsilon) \\
\text{(A.7)} \qquad &\gamma_{\underline{4}} = i\beta_4 \\
&\gamma_{\underline{2\upsilon+1}} = \beta_0 .
\end{aligned}
$$

Then the $2\upsilon + 1$ matrices γ_{α} satisfy

$$
\text{(A.8)} \qquad \gamma_{\underline{\alpha}}\gamma_{\underline{\beta}} + \gamma_{\underline{\beta}}\gamma_{\underline{\alpha}} = 2\eta_{\underline{\alpha}\underline{\beta}} = 2(\delta_{\underline{\alpha}\underline{\beta}} - 2\delta_{\underline{\alpha}}^{4}\delta_{\underline{\beta}}^{4})
$$

with $\gamma_{\underline{\alpha}}(\alpha \neq 4)$ hermitian and $\gamma_{\underline{4}}$ anti-hermitian.

It follows from equations (A.1) and (A.5) that, when the d's all differ,

$$
\text{(A.9)} \qquad \gamma_{\underline{\alpha}_1}\gamma_{\underline{\alpha}_2} \cdots \gamma_{\underline{\alpha}_{2\upsilon}} = -(i)^{\upsilon} E_{\underline{\alpha}_1 \cdots \underline{\alpha}_{2\upsilon+1}} \gamma^{\underline{\alpha}_{2\upsilon+1}}
$$

where

$$
\text{(A.10)} \qquad \gamma^{\underline{\beta}} = \eta^{\underline{\alpha}\underline{\beta}} \gamma_{\underline{\alpha}}
$$

and

$$
\text{(A.11)} \qquad \eta^{\underline{\alpha}\underline{\beta}} \eta_{\underline{\beta}\underline{\gamma}} = \delta_{\underline{\gamma}}^{\underline{\alpha}} .
$$

We define

$$
\text{(A.12)} \qquad p!\, S_{\underline{\alpha}_1 \cdots \underline{\alpha}_p} = \gamma_{\underline{\beta}_1}\gamma_{\underline{\beta}_2} \cdots \gamma_{\underline{\beta}_p} \delta_{\underline{\alpha}_1 \cdots \underline{\beta}_p}^{\underline{\beta}_1 \cdots \underline{\beta}_p}, \qquad (p = 2, 3 \ldots \upsilon).
$$

The traces of the matrices $\gamma_{\underline{\alpha}}$, $S_{\underline{\alpha}_1 \cdots \underline{\alpha}_p}$ vanish. Further $S_{\underline{\alpha}_1 \cdots \underline{\alpha}_p}$ vanishes unless all the indices α_i are different and in that case

$$
S_{\underline{\alpha}_1 \cdots \underline{\alpha}_p} = \gamma_{\underline{\alpha}_1}\gamma_{\underline{\alpha}_2} \cdots \gamma_{\underline{\alpha}_p} .
$$

It is then a consequence of the equation (A.9) that

$$
\begin{aligned}
&S_{\underline{\alpha}_1 \cdots \underline{\alpha}_{2\upsilon-2}} S_{\underline{\delta}\underline{\epsilon}} = -(i)^{\upsilon} E_{\underline{\alpha}_1 \cdots \underline{\alpha}_{2\upsilon-2}\underline{\delta}\underline{\epsilon}\underline{\gamma}} \gamma^{\underline{\gamma}} \\
\text{(A.13)} \qquad &\quad + (\eta_{\underline{\lambda}\underline{\delta}} S_{\underline{\beta}_1 \cdots \underline{\beta}_{2\upsilon-2}\underline{\epsilon}} - \eta_{\underline{\lambda}\underline{\epsilon}} S_{\underline{\beta}_1 \cdots \underline{\beta}_{2\upsilon-3}\delta}) \delta_{\underline{\alpha}_1 \cdots \underline{\alpha}_{2\upsilon-2}}^{\underline{\beta}_1 \cdots \underline{\beta}_{2\upsilon-3}\lambda} \\
&\quad + \eta_{\underline{\lambda}\underline{\delta}} \eta_{\underline{\nu}\underline{\epsilon}} S_{\underline{\beta}_1 \cdots \underline{\beta}_{2\upsilon-4}} \delta_{\underline{\alpha}_1 \cdots \underline{\alpha}_{2\upsilon-2}}^{\underline{\beta}_1 \cdots \underline{\beta}_{2\upsilon-4}\underline{\nu}\lambda} .
\end{aligned}
$$

When $\upsilon = 2$, equations (A.12) and (A.13) become

$$
\text{(A.14)} \qquad 2S_{\underline{\alpha}\underline{\beta}} = \gamma_{\underline{\alpha}}\gamma_{\underline{\beta}} - \gamma_{\underline{\beta}}\gamma_{\underline{\alpha}},
$$

and

$$
\begin{aligned}
\text{(A.15)} \qquad &S_{\underline{\alpha}\underline{\beta}} S_{\underline{\delta}\underline{\epsilon}} = E_{\underline{\alpha}\underline{\beta}\underline{\delta}\underline{\epsilon}\underline{\gamma}} \gamma^{\underline{\gamma}} + (\eta_{\underline{\alpha}\underline{\epsilon}}\eta_{\underline{\beta}\underline{\delta}} - \eta_{\underline{\beta}\underline{\epsilon}}\eta_{\underline{\alpha}\underline{\delta}})\, 1_4 \\
&\quad + \eta_{\underline{\alpha}\underline{\epsilon}} S_{\underline{\beta}\underline{\delta}} - \eta_{\underline{\beta}\underline{\epsilon}} S_{\underline{\alpha}\underline{\delta}} - \eta_{\underline{\alpha}\underline{\delta}} S_{\underline{\beta}\underline{\epsilon}} + \eta_{\underline{\beta}\underline{\delta}} S_{\underline{\alpha}\underline{\epsilon}}
\end{aligned}
$$

respectively.

It is a further consequence of equation (A.9) that

(A.16) $$E^{\underline{\alpha}_1\cdots\underline{\alpha}_{2v-2}\lambda\mu\nu}S_{\underline{\alpha}_1\cdots\underline{\alpha}_{2v-2}} = (2v-2)!\,(i)^v S^{\underline{\lambda}\underline{\mu}\underline{\nu}}.$$

Since

$$\gamma^{\underline{\lambda}}S^{\underline{\mu}\underline{\nu}} = \gamma^{\underline{\lambda}}\gamma^{\underline{\mu}}\gamma^{\underline{\nu}} - \gamma^{\underline{\lambda}}\eta^{\underline{\mu}\underline{\nu}}$$

it may be shown that

(A.17) $$S^{\underline{\lambda}\underline{\mu}\underline{\nu}} = \gamma^{\underline{\lambda}}\gamma^{\underline{\mu}}\gamma^{\underline{\nu}} - \gamma^{\underline{\lambda}}\eta^{\underline{\mu}\underline{\nu}} + \gamma^{\underline{\mu}}\eta^{\underline{\nu}\underline{\lambda}} - \gamma^{\underline{\nu}}\eta^{\underline{\mu}\underline{\lambda}}$$

and that

(A.18) $$S^{\underline{\alpha}\underline{\beta}}\gamma^{\underline{\gamma}} - \gamma^{\underline{\gamma}}S^{\underline{\alpha}\underline{\beta}} = 2(\gamma^{\underline{\alpha}}\eta^{\underline{\beta}\underline{\gamma}} - \gamma^{\underline{\beta}}\eta^{\underline{\alpha}\underline{\gamma}})$$

From formulas similar to equations (A.13) it may be verified that the trace of the products of $S_{\underline{\alpha}_1\cdots\underline{\alpha}_p}$ and $S^{\underline{\beta}_1\cdots\underline{\beta}_q}$ vanish unless $p = q$ and in that case

(A.19) $$S^{\underline{\alpha}_1\cdots\underline{\alpha}_p}S_{\underline{\beta}_1\cdots\underline{\beta}_p} = (-1)^{\frac{p}{2}(p-1)}1_{2^v}\delta^{\underline{\alpha}_1\cdots\underline{\alpha}_p}_{\underline{\beta}_1\cdots\underline{\beta}_p}.$$

An arbitrary complex $2^v \times 2^v$ matrix x may be expressed as

(A.20) $$X = y1_{2^v} + y^{\underline{\alpha}}\gamma_{\underline{\alpha}} + \sum_{p=2}^{v}\frac{1}{p!}\,y^{\underline{\alpha}_1\cdots\underline{\alpha}_p}S_{\underline{\alpha}_1\cdots\underline{\alpha}_p}$$

with

$$2^v y = \text{trace } X$$

(A.21) $$2^v y^{\underline{\alpha}} = \text{trace }(X\gamma^{\underline{\alpha}})$$

$$2^v y^{\underline{\alpha}_1\cdots\underline{\alpha}_p} = (-1)^{\frac{p}{2}(p-1)}\text{trace }(XS^{\underline{\alpha}_1\cdots\underline{\alpha}_p}).$$

On substituting these expressions for y, $y^{\underline{\alpha}}$, $y^{\underline{\alpha}_1\cdots\underline{\alpha}_b}$ into (A.2)) and using the fact that X is an arbitrary matrix we find that

(A.22) $$2^v\delta^A_C\delta^D_B = \delta^D_C\delta^A_B + \gamma^{\alpha D}{}_C\gamma_\alpha{}^A{}_B + \sum_{p=2}^{v}(-1)^{\frac{p}{2}(p-1)}S^{\alpha_1\cdots\alpha_p D}{}_C S_{\alpha_1\cdots\alpha_p}{}^A{}_B,$$

where A, B, C and D take on values from 1 to 2^v.

When the matrices $\gamma_{\underline{\alpha}}$ are given as above we find that the matrix

(A.23) $$\gamma = \rho\gamma_1\gamma_3\cdots\gamma_{2v-1}$$

where ρ is a complex number, satisfies

(A.24) $$\gamma^T = e\gamma, \quad e = (-1)^{\frac{v}{2}(v+1)}$$

and that

(A.25) $$(\gamma\gamma_\alpha)^T = f\gamma\gamma_\alpha \quad f = (-1)^v e.$$

That is when v is even, the matrices γ, $\gamma\gamma_\alpha$ are all symmetric or all antisymmetric.

When $v = 2$ we may choose γ so that γ_{AB} the components of the antisymmetric matrix γ are normalized so that

(A.26) $$\gamma_{AB}\gamma_{CD}\epsilon^{ABCD} = -8 \quad (A, B, C, D = 1, 2, 3, 4).$$

The matrix γ and its inverse with components

(A.27) $$2\gamma^{AB} = \epsilon^{ABCD}\gamma_{CD}$$

which satisfies the equation

(A.28) $$\gamma_{AB}\gamma^{BC} = \delta_A^C$$

may be used to raise and lower indices of spinors. The rules are such that in raising and lowering indices we always sum with respect to the second index on the γ's involved.

The matrices

$$\| \gamma_{\underline{\alpha}}{}^{LM} \| = \| \gamma_{\underline{\alpha}}{}^{L}{}_{N}\gamma^{MN} \| = \gamma_{\underline{\alpha}}\gamma^{-1}$$

are anti-symmetric and together with $\gamma^{-1} = \| \gamma^{LM} \|$ form a basis for all 4×4 anti--symmetric matrices $X = \| X^{LM} \| = -X^T$. Since

$$\gamma_{\underline{\alpha}}{}^{LM}\gamma_{LM} = \gamma_{\underline{\alpha}}{}^{L}{}_{N}\gamma^{MN}\gamma_{LM} = \gamma_{\underline{\alpha}}{}^{L}{}_{L} = 0$$

the quantities

$$2\gamma_{\underline{\alpha}LM} = \epsilon_{LMPQ}\gamma_{\underline{\alpha}}{}^{PQ} = 2\gamma_{LN}\gamma_{\underline{\alpha}}{}^{N}{}_{M}.$$

That is the process of lowering the indices on $\gamma_\alpha{}^{LM}$ by means of γ gives the same results as manipulating a pair of indices by means of the four index Levi--Civita anti-symmetric density. These two methods of manipulating indices were discussed by Veblen [9].

He also used the fact any anti-symmetric matrix X^{LM} may be expressed as

$$2X^{LM} = X\gamma^{LM} + X^{\underline{\alpha}}\gamma_{\underline{\alpha}}{}^{LM}$$

where

$$2X = -X^{LM}\gamma_{LM}$$

$$2X^{\underline{\alpha}} = X^{LM}\gamma^{\underline{\alpha}}{}_{LM}.$$

On substitutiong for X and X^{α} into the above equation we find that we must have

(A.29) $$\gamma^{\underline{\alpha}}{}_{LM}\gamma_{\underline{\alpha}}{}^{PQ} - \gamma_{LM}\gamma^{PQ} = 2\delta^{PQ}_{LM}.$$

This equation may be written as

(A.30) $$\gamma^{\underline{\alpha}L}{}_{M}\gamma_{\underline{\alpha}}{}^{P}{}_{Q} = 2\gamma^{LP}\gamma_{QM} + 2\delta^{L}_{Q}\delta^{P}_{M} - \delta^{L}_{M}\delta^{P}_{Q}$$

from which it follows that

(A.31) $$S^{\underline{\alpha}\underline{\beta}L}{}_{N}S_{\underline{\alpha}\underline{\beta}}{}^{P}{}_{Q} = -4(\delta^{L}_{Q}\delta^{P}_{N} + \gamma^{LP}\gamma_{NQ})$$

Equations (A.30) and (A.31) hold only in the case $v = 2$ where as equations (A.22) hold for arbitrary v.

Note that equation (A.23) defines the matrix $\gamma = \|\gamma_{AB}\|$ in a particular coordinate system in a linear space S_{2^v}. In another coordinate system obtained from this one by the transformation

$$\psi'^{A} = T^{A}{}_{B}\psi^{B}$$

we have

$$\gamma' = t^{T}\gamma t$$

where t is the matrix inverse to the matrix $T = \|T^{A}{}_{B}\|$.

In the coordinate system in which equation (A.23) holds the matrix

(A.32) $$H = \|H_{\dot{A}B}\| = \gamma_4$$

is anti-hermitian. That is

$$\overline{H}^{T} = -H.$$

The $(2v + 1)$ matrices $H\gamma_{\alpha}(\alpha = 1, 2, \ldots, 2v + 1)$ are hermitian, that is

(A.33) $$(\overline{H\gamma_{\alpha}})^{T} = H\gamma_{\alpha}.$$

It then follows that

(A.34) $$(\overline{HS_{\underline{\alpha}_1\cdots\underline{\alpha}_b}})^{T} = (-1)^{i}(HS_{\underline{\alpha}_1\cdots\underline{\alpha}_b}) = \pm HS_{\underline{\alpha}_1\cdots\underline{\alpha}_b}.$$

Thus the matrices $HS_{\underline{\alpha}_1\cdots\underline{\alpha}_b}$ are hermitian when

(A.35) $$(i = 2s) \quad b = 1 \text{ or } 2 \bmod 4$$

and are anti-hermitian when

(A.36) $$(i = 2s + 1) \quad b = 3 \text{ or } 0 \bmod 4.$$

The matrix H has the transformation law

$$H' = \bar{t}^T H t$$

and the matrices $H\gamma_{\underline{\alpha}}$, $HS_{\underline{\alpha}_1 \cdots \underline{\alpha}_p}$ have similar ones.

When $\omega^{\underline{\beta}}_{\alpha}$ are a set of five one forms in V_5 such that

(A.37) $$\eta_{\underline{\beta}\underline{\gamma}} \omega^{\underline{\beta}}_{\alpha} \omega^{\underline{\gamma}}_{\delta} = \gamma_{\alpha\delta}$$

where $\gamma_{\alpha\delta}$ is a metric tensor in $V_{2\upsilon+1}$ the $2^\upsilon \times 2^\upsilon$ matrices

(A.38) $$\gamma_\alpha = \omega^{\underline{\beta}}_{\alpha} \gamma_{\underline{\beta}}$$

satisfy equations (A.2) when $\upsilon = 2$ and the matrices

(A.39) $$\gamma^\alpha = e^{\alpha}_{\underline{\beta}} \eta^{\underline{\beta}\underline{\delta}} \gamma_{\underline{\delta}}$$

where $e^{\alpha}_{\underline{\beta}}$ are such that

(A.40) $$\omega^{\underline{\beta}}_{\alpha} e^{\delta}_{\underline{\beta}} = \delta^{\delta}_{\alpha}$$

satisfy

(A.41) $$\text{trace}\,(\gamma^\alpha \gamma_\beta) = 2^\upsilon \delta^\alpha_\beta .$$

The matrices γ_α (and γ^α) determine a basis for all $2^\upsilon \times 2^\upsilon$ matrices obtained from the basis discussed above by replacing $\gamma_{\underline{\alpha}}$ (and $\gamma^{\underline{\alpha}}$) by the former matrices.

One uses equations (A.38) and (A.39) to accomplish the replacement. It then follows that the equations given above such as equations (A.8) through (A.22) may be written as equations involving the γ_α's provided the $\eta^{\underline{\alpha}\underline{\beta}}$ (and $\eta_{\underline{\alpha}\underline{\beta}}$) are replaced by $\gamma^{\alpha\beta}$ (and $\gamma_{\alpha\beta}$) respectively.

In case $\upsilon = 2$ when the $\omega^{\underline{\beta}}_{\alpha}$ are those given in Section 2, we have

(A.42) $$\gamma_i = \tilde{\gamma}_i + WA_i\, \gamma_{\underline{5}}$$

(A.43) $$\gamma_5 = W \gamma_{\underline{5}}$$

where

(A.44) $$\tilde{\gamma}_i \tilde{\gamma}_j + \tilde{\gamma}_j \tilde{\gamma}_i = 2 g_{ij}\, 1_4$$

when

(A.45) $$\tilde{\gamma}_i = \omega^{\underline{a}}_{i} \gamma_{\underline{a}} .$$

Then

(A.46) $$\begin{aligned} \gamma^i &= \gamma^{i\alpha}\gamma_\alpha = \tilde{\gamma}^i = g^{ij}\tilde{\gamma}_j \\ \gamma^5 &= W^{-1}(\gamma_{\underline{5}} - WA^i \tilde{\gamma}_i). \end{aligned}$$

The matrices

(A.47) $$2S_{\alpha\beta} = 2\,\omega^{\gamma}_{\underline{\alpha}}\,\omega^{\delta}_{\underline{\beta}}\,S_{\underline{\gamma\delta}} = \gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha$$

are such that

(A.48) $$\begin{aligned} S_{ij} &= \tilde{S}_{ij} + WA_j\tilde{\gamma}_i\gamma_{\underline{5}} - WA_i\tilde{\gamma}_j\gamma_{\underline{5}} \\ S_{5i} &= W\gamma_{\underline{5}}\tilde{\gamma}_i \end{aligned}$$

where

(A.49) $$\begin{aligned} 2\tilde{S}_{ij} &= \tilde{\gamma}_i\tilde{\gamma}_j - \tilde{\gamma}_j\tilde{\gamma}_i \\ S^{ij} &= \tilde{S}^{ij} \\ S^{5j} &= W^{-1}\gamma_{\underline{5}}\tilde{\gamma}^j - A_i\tilde{S}^{ij}. \end{aligned}$$

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